INTEGRATED FORM OF SINGULAR MINDLIN'S SOLUTION

Francesco Basile
Halcrow Group Ltd
Vineyard House
44 Brook Green
London W6 7BY

Abstract
The derivation of the integrated form of the singular Mindlin's solution[1] over a cylindrical surface is presented. This solution is often employed in numerical analyses of piled foundations and may have other applications in geomechanics. The analytical integration of the singular Mindlin's function leads to significant savings of computational resources, especially in large pile group problems. This represents an advance over previous work where this function has been integrated numerically, and considerable computing resources are therefore needed (eg Banerjee & Driscoll[2], Poulos[3]).

Introduction

As part of the development of a numerical code for the non-linear analysis of pile groups under general loading conditions (Basile[4, 5]), it has been found useful to obtain the integrated form of the singular Mindlin's solution. Mindlin's solution is particularly convenient for analysing the pile problem since the boundary conditions along the unloaded ground surface are automatically satisfied, and therefore this solution is widespread for boundary element analyses of piled foundations (eg Banerjee & Driscoll[2], Poulos[3], Clancy & Randolph[6]).

The flexibility coefficients of Mindlin's solution express the displacement at any point (field point A) within a homogeneous, isotropic elastic half-space due to a point load (P) acting at any other point (load point B), in terms of the coordinates of the load and field points and the elastic properties of the solid, namely, the shear modulus G_s and the Poisson's ratio ν_s (refer to Fig. 1). Mindlin's solution becomes singular whenever the field point A and the load point B coincide, ie $R_1 = 0$.

For the case of a vertically loaded pile, the application of the boundary element method typically involves the discretization of the pile-soil interface into a number of shaft cylindrical elements, the definition of the pile and the surrounding soil domains separately and then compatibility and equilibrium conditions at the interface are imposed. This implies that, in the application of Mindlin's solution to define the soil domain, the field point (ie the nodal point at which the displacement is calculated) is located at the mid-height of the element along the pile-soil interface. However, since the pile is narrow compared with its length, it has been assumed that all positions refer to the mid-height of the element on the centre line of the pile. This approximation is necessary to allow the analytical integration of the singular Mindlin's function, thereby leading to significant savings in computational costs. On the other hand, if the field point is located along the pile-soil interface, the integration is only conveniently evaluated numerically (eg Poulos & Davis[7]). The error generated by this approximation is represented by the pile radius, ie the horizontal distance of the centre line of the pile from the pile shaft surface. Consequently, the effects of this approximation become less significant for slender pile elements. It has been found that the discrepancies between the two approaches become negligible for height-to-diameter ratios of the pile element greater than 2.

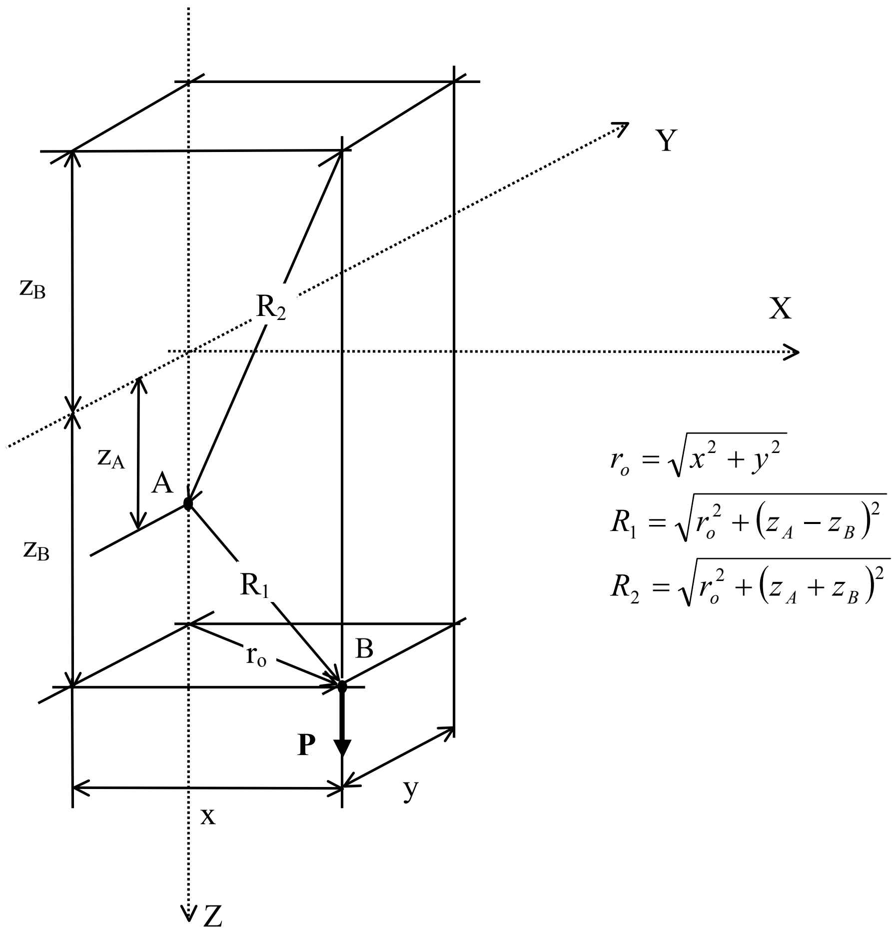

Fig. 1 Geometry for Mindlin problem

General equation and solution

The derivation of the integrated form of the singular Mindlin's solution is presented here in full. The vertical displacement w at the nodal point A caused by a vertical point load P at B may be expressed as:

$$w(A) = G(A,B)P(B) \tag{1}$$

where $G(A,B)$ is given by Mindlin's solution:

$$G(A,B) = \frac{1}{16\pi \mathrm{G_s}(1\text{-}\nu_\mathrm{s})}\left\{\frac{3-4\nu_s}{R_1} + \frac{8(1-\nu_s)^2-(3-4\nu_s)}{R_2} + \frac{(z_A-z_B)^2}{R_1^3} + \frac{(3\text{-}4\nu_s)(z_A+z_B)^2-2z_Az_B}{R_2^3} + \frac{6z_Az_B(z_A+z_B)^2}{R_2^5}\right\} \tag{2}$$

For distributed (constant) tractions t over a surface S, we obtain:

$$w(A) = t(B) \iint_S G(A,B) dS(B) \tag{3}$$

where the axial tractions t are distributed over a cylindrical shaft area of the pile-soil interface. The singular part $G^*(A,B)$ of Mindlin's solution may be expressed as:

$$G^*(A,B) = C\left[\frac{3-4\nu_s}{R_1} + \frac{(z_A - z_B)^2}{R_1^3}\right] \tag{4}$$

where C is a constant equal to $\dfrac{1}{16\pi\, G_s(1-\nu_s)}$. In order to integrate Equ. (4), it is convenient to choose the origin of coordinates at the nodal point A, ie $z_A = 0$, and, for simplicity, $z_B = z$. Therefore, R_1 may be expressed as $R_1 = \sqrt{r_o^2 + z^2}$, where r_o represents the pile radius (refer to Fig. 1). Thus, Equ. (4) becomes:

$$G^*(A,B) = C\left[\frac{3-4\nu_s}{\sqrt{r_o^2 + z^2}} + \frac{z^2}{(r_o^2 + z^2)^{3/2}}\right] \tag{5}$$

and the following surface integral has to be evaluated:

$$\iint_S G^*(A,B) dS(B) = C\iint_S \left[\frac{3-4\nu_s}{\sqrt{r_o^2 + z^2}} + \frac{z^2}{(r_o^2 + z^2)^{3/2}}\right] dS(B) \tag{6}$$

where S is the cylindrical surface of the pile shaft element. The two integrals in Equ. (6) may be evaluated (in polar coordinates) separately. The first integral yields:

$$(3-4\nu_s)\iint_S \frac{1}{\sqrt{r_o^2+z^2}} dS(B) = (3-4\nu_s)\int_0^{2\pi}\int_{-h/2}^{h/2} \frac{1}{\sqrt{r_o^2+z^2}} r_o dz d\vartheta = 2\pi\, r_o(3-4\nu_s)\left[\ln\left|z+\sqrt{z^2+r_o^2}\right|\right]_{-h/2}^{h/2} =$$

$$2\pi\, r_o(3-4\nu_s)\ln\frac{h+\sqrt{h^2+4r_o^2}}{-h+\sqrt{h^2+4r_o^2}} = \pi\, d(3-4\nu_s)\ln\frac{h+\sqrt{h^2+d^2}}{-h+\sqrt{h^2+d^2}}$$

where h is the height of the pile shaft element and d is the pile shaft diameter. The second integral yields:

$$\iint_S \frac{z^2}{(r_o^2+z^2)^{3/2}} dS(B) = \int_0^{2\pi}\int_{-h/2}^{h/2} \frac{z^2}{(r_o^2+z^2)^{3/2}} r_o dz d\vartheta = 2\pi\, r_o \int_{-h/2}^{h/2} \frac{z^2}{(r_o^2+z^2)^{3/2}} dz$$

This integral may be evaluated by substitution — if we let $z = r_o \tan\vartheta$, then:

$$dz = r_o \sec^2 \vartheta \; d\vartheta$$

Thus:

$$\int_{-h/2}^{h/2} \frac{z^2}{\left(r_o^2 + z^2\right)^{3/2}} dz = \int_{-\arctan h/2r_o}^{\arctan h/2r_o} \frac{r_o^2 \tan^2 \vartheta}{\left(r_o^2 \tan^2 \vartheta + r_o^2\right)^{3/2}} r_o \sec^2 \vartheta \; d\vartheta = \int_{-\arctan h/d}^{\arctan h/d} \tan \vartheta \sin \vartheta \, d\vartheta$$

The above integral may be evaluated by parts — we let:

$$u = \tan \vartheta, \quad dv = \sin \vartheta \, d\vartheta, \quad du = \sec^2 \vartheta \; d\vartheta, \quad v = -\cos \vartheta$$

Thus:

$$\int_{-\arctan h/d}^{\arctan h/d} \tan \vartheta \sin \vartheta \, d\vartheta = \left[-\cos \vartheta \; \tan \vartheta\right]_{\text{-arctan h/d}}^{\text{arctan h/d}} + \int_{-\arctan h/d}^{\arctan h/d} \cos \vartheta \sec^2 \vartheta \, d\vartheta =$$

$$\left[-\sin \vartheta\right]_{-\arctan h/d}^{\arctan h/d} + \int_{-\arctan h/d}^{\arctan h/d} \sec \vartheta \, d\vartheta = -2 \sin \arctan \frac{h}{d} + \ln \frac{\sec \arctan \frac{h}{d} + \frac{h}{d}}{\sec \arctan \frac{h}{d} - \frac{h}{d}}$$

Thus, the final expression of Equ. (6) is:

$$\iint_S G^*(A,B) dS(B) = C\pi \, d \left[(3 - 4\nu_s) \ln \frac{h + \sqrt{h^2 + d^2}}{-h + \sqrt{h^2 + d^2}} - 2 \sin \arctan \frac{h}{d} + \ln \frac{\sec \arctan \frac{h}{d} + \frac{h}{d}}{\sec \arctan \frac{h}{d} - \frac{h}{d}} \right]$$

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